Unit 4

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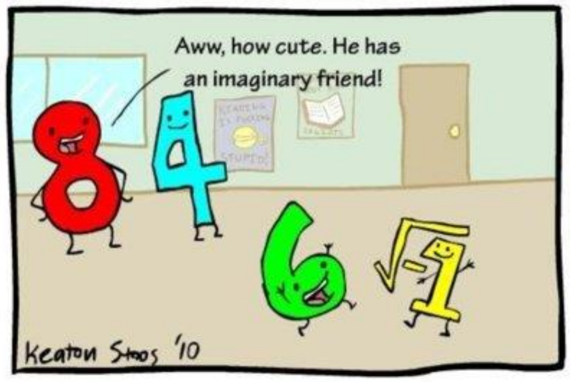
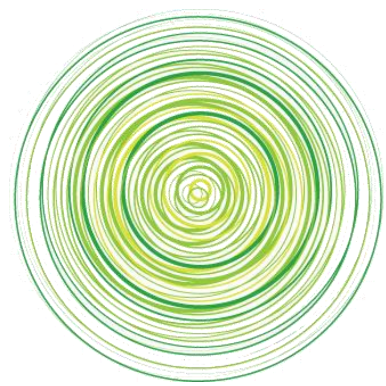
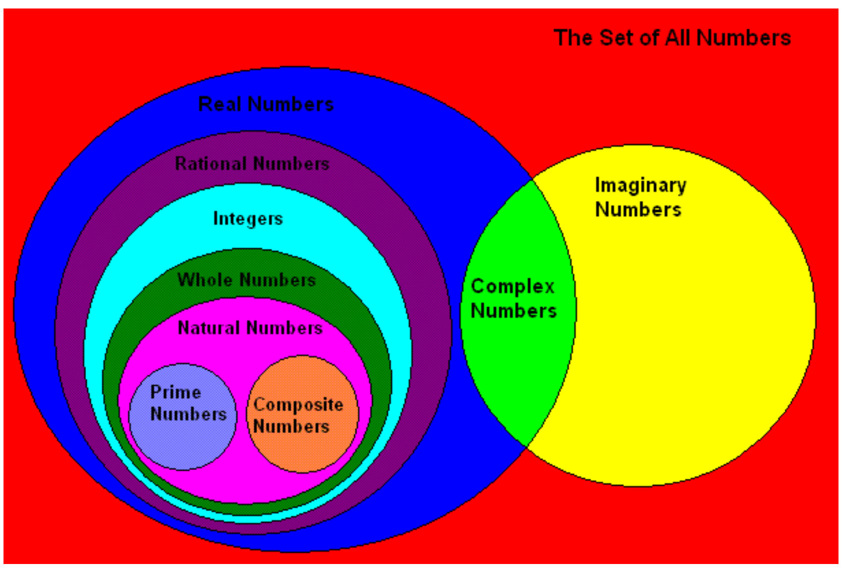
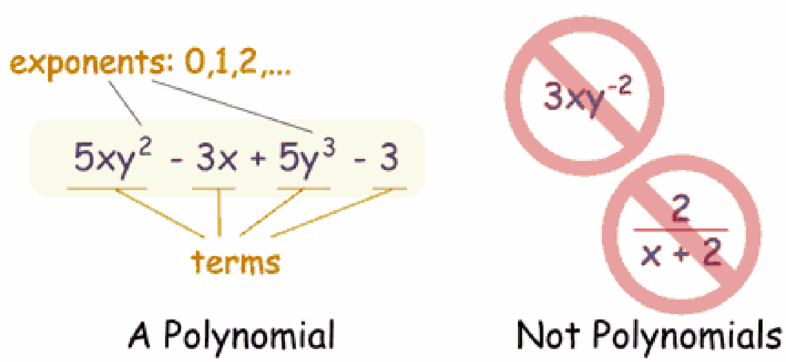




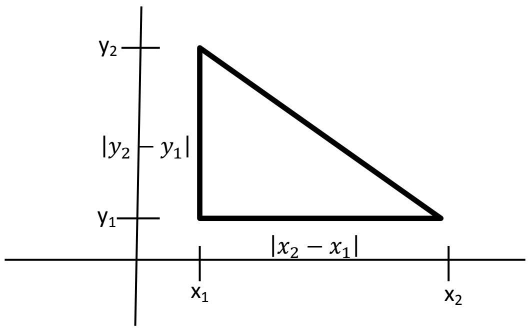




4.1 Distance and Midpoints

The Distance Formula

The distance between two points in the Cartesian coordinate system can be calculated using the Pythagorean Theorem. Recall that the Pythagorean Theorem states that for a right triangle with legs of lengths a and b and a hypotenuse of length c then $a^2 + b^2 = c^2$. Look at the triangle shown below.



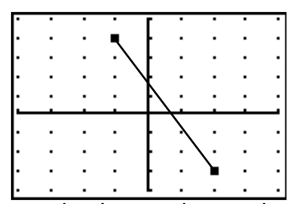
The lengths of the legs are $|x_2 - x_1|$ and $|y_2 - y_1|$. The distance between the points (x_1, y_1) and (x_2, y_2) is the hypotenuse of the triangle. Applying the Pythagorean Theorem, $d^2 = (x_2 - x_1)^2 - (y_2 - y_1)^2$. Since distance is non-negative, then the distance, d, between the points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

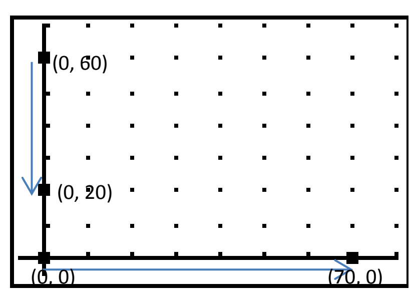
Examples:

1. Find the length of the line segment between the points (-1, 4) and (2, -3).

   $d = \sqrt{(2-(-1))^2 + (-3-4)^2} = \sqrt{9+49} = \sqrt{58} \approx 7.62$



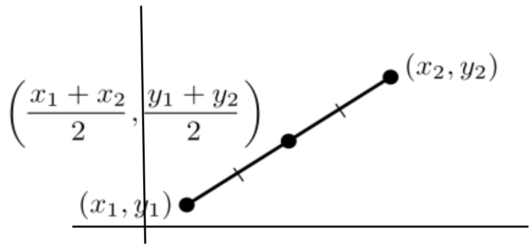
2. Suppose that at noon car A is traveling south at 20 miles per hour and is located 60 miles north of car B. Car B is traveling east at 35 miles per hour. Let (0, 0) be the coordinates of car B where the units are in miles. Plot the location of each car at 2:00 PM and find the distance between the cars at this time.



   Car B started at (0, 0) and traveled at 35 miles per hour for two hours. So it traveled a distance of 70 miles. Car A started 60 miles north or at the point (0, 60) and traveled south at 20 miles per hour for two hours for a distance of 40 miles. Thus, its new coordinate is (0, 20). To find the distance between the two cars, we use the distance formula. $d = \sqrt{(70-0)^2 + (0-20)^2} = \sqrt{4900+400} = \sqrt{5300} \approx 72.8\ miles.$

The Midpoint Formula

If a line segment is drawn between two points, then its midpoint is the point on the line segment that is equidistant from the endpoints. On a number line, the midpoint between two points is the average of the two coordinates. The midpoint in the Cartesian coordinate system is similar to the formula for the midpoint on a number line except that both coordinates are averaged. Thus, the midpoint between points (x_1, y_1) and (x_2, y_2) is the point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. See the picture below.



Example:

If the population of the United States was 203 million in 1970 and in 1990 the population was 249 million, use the midpoint formula to estimate the population in 1980.

We are given two points with this data. The points are (1970, 203) and (1990, 249). The midpoint of these two points is:

$$\left(\frac{1970+1990}{2}, \frac{203+249}{2}\right) = (1980, 226)$$

Therefore, the estimated population is 1980 is 226 million.

Homework:

1. Find the distance between the points (-4, 5) and (0, 7).
2. Find the distance between the points (1, -3) and (-2, 6).
3. Find the distance between the points (2.5, 3) and (1.75, 3.2).
4. Find the distance between the points (-0.1, 8.1) and (6.3, 9.2).
5. An isosceles triangle has at least two sides of equal length. Determine whether the triangle with vertices (0, 0), (3, 4) and (7, 1) is isosceles.

6. At 10:00 AM, car A is traveling north at 40 miles per hour and is located 50 miles south of car B. Car B is traveling west at 30 miles per hour.

   A. Let (0, 0) be the initial coordinates of car B, where the units are in miles. Plot the locations of each car at 10:00 AM and 12:30AM.

   B. Find the distance between the cars at 12:30 AM.

   C. Find the midpoint between the two cars at 12:30 AM.

7. Find the midpoint between the points (9, 4) and (-1, 2).

8. Find the midpoint between the points (-2, 7) and (-6, -2).

9. In the year 1983, the average daily jail population was 228 thousand people. In 1993, the average daily jail population had risen to 466 thousand people. Find the midpoint and explain what it represents in context.

10. In the year 1987, the federal debt was 2354 billion dollars. This had risen to 2881 billion dollars in 1989. Find the midpoint and explain what it represents in context.

11. Find the equation of a circle with a center of (8, -2) which has the point (-2, 5) on the circle.

12. Find the equation of a circle which has the endpoints of a diameter of the circle at (-5, -7) and (1, 3).

4.2 Laws of Exponents

This section should be a review of the rules for performing operations on variables with exponents.

Exponent Law	**Example**
1. $x^1 = x$	$6^1 = 6$
2. $x^0 = 1$	$7^0 = 1$
3. $x^{-1} = 1/x$	$4^{-1} = 1/4$
4. $x^m x^n = x^{m+n}$	$x^2 x^3 = x^{2+3} = x^5$
5. $x^m/x^n = x^{m-n}$	$x^6/x^2 = x^{6-2} = x^4$
6. $(x^m)^n = x^{mn}$	$(x^2)^3 = x^{2\times3} = x^6$
7. $(xy)^n = x^n y^n$	$(xy)^3 = x^3 y^3$
8. $(x/y)^n = x^n/y^n$	$(x/y)^2 = x^2 / y^2$
9. $x^{-n} = 1/x^n$	$x^{-3} = 1/x^3$

The first three laws above ($x^1 = x$, $x^0 = 1$ and $x^{-1} = 1/x$) are just part of the natural sequence of exponents.

Example: Powers of 5

$\mathbf{5^2}$	**1 × 5 × 5**	25
$\mathbf{5^1}$	**1 × 5**	5
$\mathbf{5^0}$	**1**	1
$\mathbf{5^{-1}}$	**1 ÷ 5**	$\frac{1}{5} = 0.2$
$\mathbf{5^{-2}}$	**1 ÷ 5 ÷ 5**	$\frac{1}{5^2} = \frac{1}{25} = 0.04$
	.. etc..	

Look at the table for a while ... notice that positive, zero or negative exponents are really part of the same pattern, i.e. 5 times larger (or 5 times smaller) depending on whether the exponent gets larger (or smaller).

Negative Exponent Rule: A negative exponent means division. To rewrite a quantity with a negative exponent, create a fraction with the quantity in the denominator with a positive exponent. It is important to note that if a negative exponent already appears in the denominator of a fraction, then it will move to the numerator as a positive exponent. In short, a negative exponent changes the location (numerator or denominator) of an expression and changes the sign of the exponent. $x^{-n} = 1/x^n$

Examples:

1. $6^{-3} = \frac{1}{6^3} = \frac{1}{216}$
2. $t^{-5} = \frac{1}{t^5}$
3. $\frac{1}{h^{-8}} = h^8$

Product Rule: To multiply two powers with the same base, add the exponents and leave the base unchanged. $x^m x^n = x^{m+n}$

Examples:

1. $x^6x^2 = x^8$
2. $x^{-2}x^5 = x^3$
3. $y^{-4}y^{-2} = y^{-6} = \frac{1}{y^6}$

Quotient Rule: To divide two powers with the same base, subtract the exponents and leave the base unchanged. $x^m/x^n = x^{m-n}$

Examples:

1. $\frac{n^5}{n^2} = n^3$
2. $\frac{t^3}{t^4} = \frac{1}{t} = t^{-1}$
3. $\frac{d^4}{d^{-2}} = d^{4-(-2)} = d^6$

Power to a Power Rule: To raise a power to a power, keep the same base and multiply the exponents. $(x^m)^n = x^{mn}$

Examples:

1. $(c^4)^7 = c^{28}$
2. $(x^{-3})^5 = x^{-15}$
3. $(w^{-2})^{-4} = w^8$

Power of a Product Rule: The power of a product is equal to the product of the powers of each of its factors. $(xy)^n = x^ny^n$

Examples:

1. $(xy)^2 = x^2y^2$
2. $(4y)^3 = 64y^3$
3. $(mn^2)^5 = m^5n^{10}$

Power of a Quotient Rule: The power of a quotient is equal to the quotient of the powers of the numerator and denominator. $(x/y)^n = x^n/y^n$

Examples:

1. $(x/y)^5 = x^5/y^5$
2. $(x^2/y)^3 = x^6/y^3$
3. $(3/y)^{-2} = 3^{-2}/y^{-2} = y^2/3^2 = y^2/9$

Homework:

Simplify by applying the appropriate laws of exponents.

1. x^9x^3

2. $x^{-4}x^2\,y^3y^8$

3. $5^2(5^4)$

4. $(m^2n^5)(mn^{-2})$

5. $\frac{d^2}{d^4}$

6. $\frac{h^{-3}}{h^7}$

7. $\frac{8c^2}{4c^{-9}}$

8. $\frac{u^5}{u}\frac{v^3}{v^2}$

9. $(w^9)^2$

10. $(x^{-4})^{-5}$

11. $(3y)^3$

12. $(-2x^6y^2)^3$

13. $\frac{x^3y^{-4}}{xy^5}$

14. $\frac{(5x)^2}{5^{-1}x^4}$

15. $\frac{2x^{-5}y}{(6y)^2}$

16. $(x^2 \cdot y^{-3})^2$

17. $(3d^{-4})^2\,(2d)^3$

18. $\left(\frac{x^{-2}}{4y^3}\right)^{-3}$

19. $\frac{xy^3}{y^{-2}}\left(\frac{y}{xy^5}\right)$

20. $(8x^4y^{-3})^{-2}$

21. $\frac{-12rz^6}{20rz}$

22. $\left(\frac{-2g}{3h}\right)^3\left(\frac{g^3h^7}{gh^2}\right)^{-1}$

4.3 Fractional Exponents and Radicals

The symbol $\sqrt[n]{b}$ is used to represent the n^{th} root of a number b. To find the n^{th} root, we need to find a number raised to the n^{th} power that is equal to b. For example, $\sqrt[4]{16} = 2$ since $2^4 = 16$. The expression $\sqrt[n]{b}$ is called a radical, b is called the radicand, and n is called the index.

Examples:

1. Simplify: $\sqrt[3]{-64}$

   $\sqrt[3]{-64} = -4$ since $(-4)^3 = -64$.

2. Simplify: $\sqrt[5]{7776}$

   $\sqrt[5]{7776} = 6$ because $6^5 = 7776$

3. Approximate: $\sqrt[7]{259}$

   $\sqrt[7]{259}$ is not an integer so we use the calculator to approximate the value. $\sqrt[7]{259} \approx 2.212$

Exponent Rule: A radical can be written as a fractional exponent. $\sqrt[n]{x} = x^{1/n}$

Examples:

1. Rewrite $\sqrt[6]{t}$ using exponent notation.

   $\sqrt[6]{t} = t^{1/6}$

2. Rewrite $\frac{1}{\sqrt[3]{x}}$ using exponent notation.

   $\frac{1}{\sqrt[3]{x}} = x^{-1/3}$

Exponent Rule: $\sqrt[n]{x^m} = x^{m/n}$

Examples:

1. Rewrite the following using exponents.

   A. $\sqrt[4]{d^3} = d^{3/4}$

   B. $\sqrt[3]{x^7} = x^{7/3}$

2. Rewrite the following in radical form.

   A. $y^{2/5} = \sqrt[5]{y^2}$

   B. $6^{4/3} = \sqrt[3]{6^4} = \sqrt[3]{1296}$

All exponent rules learned previously apply to fractional exponents.

Homework:

Simplify the following:

1a. $8^{1/3}$

1b. $25^{-1/2}$

2a. 3^0

2b. 3^{-2}

3a. $4^{1/2}$

3b. $(-4)^{-3/2}$

4a. $100^{-1/2}$

4b. $27^{4/3}$

Rewrite the following in exponent notation.

5a. $\sqrt[3]{h}$

5b. $\sqrt[2]{p}$

6a. $\sqrt[4]{m^5}$

6b. $\sqrt{w^6}$

7a. $\frac{1}{\sqrt[8]{x}}$

7b. $\frac{1}{\sqrt[7]{x^2}}$

Rewrite in radical notation.

8a. $x^{2/3}$

8b. $x^{4/7}$

9a. $y^{-1/2}$

9b. $t^{-3/5}$

10a. $n^{1/4}$

10b. $c^{5/2}$

11a. $(2h)^{3/4}$

11b. $4x^{5/2}$

Perform the given operations to simplify.

12a. $\dfrac{x^7 y^{-4/3}}{xy^2}$

12b. $\dfrac{5x^{-2/3}}{5^{-1/4} x^4}$

13a. $\left(x^{12} \cdot y^{-4}\right)^{1/2}$

13b. $(5cd^{-2/5})^3$

14a. $\left(\dfrac{x^{-2/3}}{2y^3}\right)^{-3}$

14b. $\left(8x^{1/2}y^{4/3}\right)^{-1/3}$

15a. $\left(x^{2/3} \cdot y^{-3/4}\right)^2$

15b. $\dfrac{2x^{-5/2}y}{(16y)^{1/2}}$

16a. $(9c^2d^{-3})^{1/2}$

16b. $(4g^{2/3}h^{1/6})^3$

17a. $\left(x^{2/3} \cdot y^{-3/4}\right)\left(x^{2/3}y^{1/4}\right)$

17b. $\dfrac{2x^{-1/5}y^{1/2}}{(6xy)^2}$

18a. $(3d^{-2/5})^2$

18b. $\left(-\dfrac{x^{-2}}{64y^3}\right)^{-1/3}$

19a. $\dfrac{x^{\frac{1}{2}}}{y^{-2}}\left(\dfrac{yx^{-\frac{2}{3}}}{xy^5}\right)$

19b. $\dfrac{x^{\frac{1}{3}}}{y^{-2}}\left(\dfrac{xy^{-\frac{2}{3}}}{xy^{-5}}\right)$

4.4 Operations with Radicals

Because $x^{1/n} = \sqrt[n]{x}$, we can use the laws of exponents to simplify radicals.

Multiplying and dividing radicals:

Properties of Radicals

1. $\sqrt[n]{ab} = \sqrt[n]{a}\ \sqrt[n]{b}$ for a ≥ 0, b ≥ 0
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ for a ≥ 0, b > 0

Examples:

1. Simplify $\sqrt{50}$.

   $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$

2. Simplify $\sqrt[3]{\frac{27}{8}} = \frac{\sqrt[3]{27}}{\sqrt[3]{8}} = \frac{3}{2}$

When simplifying radicands containing variables with exponents, it may be helpful to remember that the radical can be written as an exponent.

Examples:

1. Simplify: $\sqrt[4]{x^{20}}$

   $\sqrt[4]{x^{20}} = x^{20/4} = x^5$

2. Simplify: $\sqrt[3]{y^6}$

   $\sqrt[3]{y^6} = y^{6/3} = y^2$

3. Simplify: $\sqrt[5]{y^8}$

   $\sqrt[5]{y^8} = \sqrt[5]{y^5 \cdot y^3} = y\ \sqrt[5]{y^3}$

   or $\sqrt[5]{y^8} = y^{8/5} = y^{1\frac{3}{5}} = y\ \sqrt[5]{y^3}$

These properties can also be used to simplify products and quotients of radicals. First apply the appropriate property to write a single radical and then simplify if possible.

Examples:

1. Simplify: $\sqrt[3]{x^2y} \cdot \sqrt[3]{x^2y^2}$

   First multiply the radicals and then simplify the result. $\sqrt[3]{x^2y} \cdot \sqrt[3]{x^2y^2} = \sqrt[3]{x^4y^3} = xy\sqrt[3]{x}$

2. Simplify: $\frac{\sqrt[5]{64x^{12}}}{\sqrt[5]{2x}}$

   First combine into one radical and then simplify the fraction. Simplify the result if possible.

   $\frac{\sqrt[5]{64x^{12}}}{\sqrt[5]{2x}} = \sqrt[5]{\frac{64x^{12}}{2x}} = \sqrt[5]{32x^{11}} = 2x^2\sqrt[5]{x}$

Sums and differences of radicals:

In order to add or subtract radicals, the radicals must have the same index and same radicand. In other words, we can only add and subtract *like radicals*.

Examples:

1. Add: $3\sqrt{x} + 5\sqrt{x}$

   $3\sqrt{x} + 5\sqrt{x} = 8\sqrt{x}$

2. Subtract: $11\sqrt[3]{y} - \sqrt[3]{y}$

   $11\sqrt[3]{y} - \sqrt[3]{y} = 10\sqrt[3]{y}$

3. Add: $\sqrt{d} + 4\sqrt[5]{d}$

   These can not be added because they are not like radicals (the indexes are different).

Rationalizing denominators:

The process of rationalizing the denominator is used to write a fraction without a radical in the denominator. To rationalize, we multiply the numerator and denominator by a radical which will give a radical in the denominator that can be simplified so that there are no radicals.

Example:

Rationalize the denominator of each fraction.

A) $\frac{1}{\sqrt{2}}$ B) $\frac{3}{\sqrt{8}}$ C) $\frac{\sqrt[3]{5}}{\sqrt[3]{9}}$

A) To rationalize this fraction, we can multiply both the numerator and denominator by $\sqrt{2}$ to get $\sqrt{4} = 2$ in the denominator. $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

B) To rationalize this fraction, we can multiply both the numerator and denominator by $\sqrt{2}$ to get $\sqrt{16} = 4$ in the denominator. $\frac{3}{\sqrt{8}} = \frac{3}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{4}$

C) To rationalize this fraction, we can multiply both the numerator and denominator by $\sqrt[3]{3}$ to get $\sqrt[3]{27} = 3$ in the denominator. $\frac{\sqrt[3]{5}}{\sqrt[3]{9}} = \frac{\sqrt[3]{5}}{\sqrt[3]{9}} \cdot \frac{\sqrt[3]{3}}{\sqrt[3]{3}} = \frac{\sqrt[3]{15}}{3}$

Homework:

Simplify the following:

1. $\sqrt{m^8}$
2. $\sqrt{4y^7}$
3. $\sqrt{x^4 y^{10}}$
4. $\sqrt{18x^6}$
5. $\sqrt[3]{x^{12}}$
6. $\sqrt[3]{y^{16}}$
7. $\sqrt[3]{16m^9}$
8. $\sqrt[3]{x^8 y^3 z}$
9. $\sqrt{64\text{y}^{18}}$
10. $\sqrt[3]{75x^6}$
11. $\sqrt{180\text{x}^5\text{y}^8}$
12. $\sqrt[3]{54\text{b}^5}$
13. $\sqrt{48\text{x}^6\text{y}^3}$
14. $\sqrt[4]{16\text{g}^{20}\text{h}^8}$
15. $\sqrt[4]{m^{12}n^4}$
16. $\sqrt[4]{32x^5 y^7}$
17. $\sqrt[5]{g^{20}h^7}$
18. $\sqrt[6]{g^{15}h^{12}}$

Perform the indicated operation and simplify if possible.

19. $7\sqrt{10}-\sqrt{10}$

20. $\sqrt{\frac{50}{16}}$

21. $3\sqrt{t}+5\sqrt{t}$

22. $8\sqrt{5}-2\sqrt[3]{5}$

23. $\sqrt[3]{3x^4y}\cdot\sqrt[3]{9x^4y^2}$

24. $\frac{\sqrt{45x^7}}{\sqrt{5x^2}}$

25. $\sqrt[4]{32x}-8\sqrt[4]{2x}$

26. $\sqrt{3x^3y}\cdot\sqrt{6x^4y}$

Approximate using your calculator.

27. $\sqrt{\frac{2}{5}}$

28. $\sqrt[6]{4130}$

Rationalize the denominator.

29. $\sqrt{\frac{2}{5}}$

30. $\frac{3}{\sqrt{7}}$

31. $\frac{4}{\sqrt[3]{4}}$

32. $\frac{\sqrt[3]{3}}{\sqrt[3]{5}}$

33. Complete the table. The speed s (in miles per hour) a car was traveling if it skidded d feet after the brakes were applied on a dry concrete road is given by s = $\sqrt{24d}$. Approximate the speed to the nearest mile per hour.

A. Fill in the table below.

Skid distance, d	Speed, s
23	
50	
75	

B. If the car skidded 45 feet, how fast was it traveling?

C. If the car was traveling at 65 mph, how far would it skid?

4.5 Solving equations with exponents and radicals

To solve equations with a radical of index n, isolate the radical on one side of the equation and then raise both sides of the equation to the n^{th} power. Note that when solving equations with radicals and exponents you should always check your answers because you can find extraneous solutions. Extraneous solutions are solutions which do not solve the original equation.

Examples:

1. Solve: $\sqrt{2x+1} - 7 = -3$

   First add 7 to both sides of the equation to isolate the radical.
   $\sqrt{2x+1} = 4$
   Since the index on a square root is 2, raise both sides to the 2^{nd} power.
   $\left(\sqrt{2x+1}\right)^2 = 4^2$
   2x + 1 = 16
   2x = 15
   x = 15/2 Answer checks in the original equation.

2. Solve: $2\sqrt[3]{x} + 5 = 9$

   First isolate the radical.
   $2\sqrt[3]{x} = 4$ Subtract 5 from both sides
   $\sqrt[3]{x} = 2$ Divide both sides by 2
   Then raise both sides to the 3^{rd} power.
   $\left(\sqrt[3]{x}\right)^3 = 2^3$
   x = 8 Answer checks in the original equation

3. Solve: $3\sqrt[4]{x+1} = -15$
   First isolate the radical.
   $\sqrt[4]{x+1} = -5$ Divide both sides of the equation by 3
   $\left(\sqrt[4]{x+1}\right)^4 = (-5)^4$ Raise both sides to the 4^{th} power
   x + 1 = 625
   x = 624
   Checking the answer in the original equation, we find $3\sqrt[4]{624+1} = -15$ which gives 15 = -15 which is not true. Therefore, x = 624 is an extraneous solution and this equation has no possible solutions.

To solve an equation involving variables with exponents, isolate the term with the exponent and then either:

1. If the exponent is n, take the n^{th} root of both sides of the equation.

or 2. Raise both sides of the equation to the reciprocal power.

Examples:

1. Solve: $3x^4 - 2 = 25$

   First isolate the x^4, then we will take the 4^{th} root of both sides.

$3x^4 = 27$	Add 2 to both sides of the equation
$x^4 = 9$	Divide both sides by 3
$x = \sqrt[4]{9} = 1.732$	Take the 4^{th} root of both sides of the equation and check the solution.

2. Solve: $(x + 3)^{2/3} - 9 = 27$

   First isolate the term with the exponent, then raise both sides to the reciprocal of the exponent which is 3/2.

$(x + 3)^{2/3} = 36$	Add 9 to both sides of the equation
$((x + 3)^{2/3})^{3/2} = 36^{3/2}$	Raise both sides to the 3/2 power.
$x + 3 = 216$	Simplify both sides
$x = 213$	Add 3 to both sides of the equation and check the solution.

Homework:

Solve the following equations.

1. $4\sqrt{x} - 6 = 14$
2. $\sqrt{x - 5} = 7$
3. $3\sqrt{4x + 3} + 1 = 19$
4. $5\sqrt{3x - 2} = -20$
5. $\sqrt[3]{t} = -2$
6. $5\sqrt[3]{m - 6} = -20$
7. $-3\sqrt[3]{2x + 5} + 6 = 18$
8. $2 - 4\sqrt[3]{8x} = -14$
9. $\sqrt[5]{d + 2} = 3$
10. $4\sqrt[7]{x + 30} = -8$
11. $\sqrt[6]{2x} + 2 = -3$
12. $3 + 2\sqrt[4]{b - 5} = 11$
13. $3x^4 - 12 = 15$
14. $5x^2 + 2 = 48$

15. $t^{-2} + 5 = 9$

16. $2w^{-3} + 8 = 32$

17. $t^{1/2} - 4 = 16$

18. $4x^{5/2} = 12$

19. $(h + 1)^{2/3} = 49$

20. $(2d + 3)^{3/5} = -6$

21. $2(x - 5)^{-1/2} = 10$

22. $3x^{-3/2} = -24$

23. During a flu epidemic in a small town, health officials estimate that the number of people infected t days after the first case was discovered is given by $P(t) = 40t^{4/7}$.
   A) How many people will be infected after 7 days?
   B) How long will it take for 250 people to be infected?

24. If you are flying in an airplane at an altitude of h miles, on a clear day you can see a distance of d miles to the horizon, where $d = \sqrt{7920h}$.
   A) At what altitude will you be able to see for a distance of 150 miles?
   B) If you are 3 miles high, how far can you see to the horizon?

25. The period of a pendulum is the time it takes for the pendulum to complete one entire swing, from left to right and back again. The greater the length, L, of the pendulum, the longer its period, T. If L is measured in feet, then the period (in seconds) is given by $T = 2\pi\sqrt{\frac{L}{32}}$.
   A) The pendulum in the United Nations building in New York is 75 feet long. What is its period?
   B) The pendulum in the math and physics building at UCF (which is unoperational) is approximately 45 feet long. What would be its period?
   C) If a pendulum has a period of 3 seconds, how long is it?

26. If you walk in a normal way, your maximum speed, v, in meters per second, is limited by the length of your legs, L (in meters), according to the formula $v = \sqrt{9.8L}$.
   A) If a typical 4-year old has legs that are 0.5 meters long, how fast can he walk?
   B) If an adult female has legs about 0.85 meters long, how fast can she walk?
   C) If a person can walk at 2.7 meters per second, how long is the person's legs?

27. Due to improvements in technology, the annual electricity cost of running a refrigerator has decreased since 1970. The average annual cost in dollars is modeled by the function $c(t) = 148 - 28t^{1/3}$, where t is the number of years since 1970.
   A) How much did it cost to run a refrigerator in 1970?
   B) How much did it cost to run a refrigerator in 1990?
   C) How much did it cost to run a refrigerator in 2012?
   D) When will the cost to run a refrigerator be half as much as it was in 1970?
   E) When will it cost $60 to run a refrigerator?

28. Small animals can not survive long without eating. The weight of a typical vampire bat, W (in grams), decreases over time until its next meal. It's weight can be modeled by $W(h) = 130.25h^{-0.126}$, where h is the number of hours since the bat's most recent meal.

A) Sketch the graph of the weight of the bat over a 72-hour time period.
B) What is the weight of the bat after 24 hours?
C) How long will it take for the weight of the bat to decrease to 90 grams?
D) The bat will reach the point of starvation if its weight drops to 78 grams. How long can the bat survive after eating until its next meal?

29. An animal's heart rate is related to its size or mass. The heart rates of mammals are given approximately by the function $H(m) = km^{-1/4}$, where m is the mass of the mammal and k is a constant. A rabbit that has a mass of 2 kg has a heart rate of 170.

A) Find the value of the constant, k.
B) Write a formula for H(m) using the k found in part A.
C) Sketch a graph of h(m) for masses up to 5000 kg.
D) Find the heart rate of a polar bear with a mass of 650 kg.
E) Find the heart rate of a human male who has a mass of 80 kilograms.
F) What is the mass of a mammal who has a heart rate of 220bpm?

30. In the sport of crew, the best times vary with the number of men in the crew, according to the formula $t=kn^{-1/9}$ where n is the number of men in the crew and t is the winning time, in minutes, for a 2000-meter race.

A) If the winning time for an 8-man crew was 5.73, find the value of k.
B) Complete the table of values of predicted winning times for other racing crews.

Size of crew, n	1	2	4	6
Winning time, t				

31. When a layer of ice forms on a pond, the thickness of the ice, d (in centimeters), varies directly as the square root of time (in minutes). The table shows the ice thickness at certain times.

t (min)	10	30	40	60
d (cm)	0.5	0.87	1.01	1.24

A) Find the constant of variation k and write a power function that relates the thickness of the ice to the time that has passed.
B) How thick was the ice after 3 hours?
C) If the thickness of the ice was 2.5 cm, how long was the ice forming?

4.6 Polynomials

A **polynomial** is either zero or can be written as the sum of a finite number of non-zero terms. Each **term** consists of the product of a constant and a finite number of variables raised to whole number powers. The exponent on a variable in a term is called the degree **of that variable** in that term; the **degree of the term** is the sum of the degrees of the variables in that term, and **the degree of a polynomial** is the largest degree of any one term in the polynomial.

A term with no variables is called a constant term, or just a constant; the degree of a (nonzero) constant term is 0.

The **coefficient** of a term is the constant (number) part of the term.

For example, these are polynomials:

- $3x$
- $-6y^2 – ½ x$
- $3xyz + xy^2z - 0.1xz - 200y + 0.5$
- $512v^5 + 99w^5$
- 11

These are not polynomials:

- 2/(x+2) is not, because variables are not allowed in denominators
- $3xy^{-2}$ is not, because the exponent is "-2" which is not a whole number
- $\sqrt{x} + 5x$ is not, because $\sqrt{x} = x^{1/2}$ which has an exponent which is not a whole number

Let's look at a few polynomials and identify the terms, coefficients, and degrees.

Example:

For each of the following polynomials, identify the terms of the polynomial, the coefficient of each term, the degree of each term and the degree of the polynomial.

A. 12
B. $½ x^7y^2$
C. $4x^2 + 8x – 5$
D. $17x^3y^2 + 4xy^3 - 0.7y$

Polynomial	Terms	Coefficients	Degree of each term	Degree of the polynomial
12	12	12	0	0
$½ x^7y^2$	$½ x^7y^2$	½	9	9
$4x^2 + 8x – 5$	$4x^2 + 8x – 5$	4, 8, -5	2, 1, 0	2
$17x^3y^2 + 4xy^3 - 0.7y$	$17x^3y^2 + 4xy^3 - 0.7y$	17, 4, -0.7	5, 4, 1	5

A **monomial** is a polynomial which has exactly one term such as 6 or $4x^3y^8$. A **binomial** has two terms. For instance, $8x^5 – 9x^3$ is a binomial. A **trinomial** has three terms such as $-3x^4 + 3x^2 – 11$. All of these are also polynomials because polynomials can have any number of terms.

The **standard form** for writing a polynomial is to put the terms with the highest degree first followed by terms in decreasing order.

Example:

Put the polynomial in standard form: $3x^2 - 7 + 4x^3 + x^6$

Solution: The highest degree is 6, so that goes first, then 3, 2 and then the constant last:

$$x^6 + 4x^3 + 3x^2 – 7$$

Polynomials are also sometimes named for their degree:

- a second-degree polynomial, such as $4x^2$, $x^2 – 9$, or $ax^2 + bx + c$, is also called a "quadratic"
- a third-degree polynomial, such as $–6x^3$ or $x^3 – 27$, is also called a "cubic"
- a fourth-degree polynomial, such as x^4 or $2x^4 – 3x^2 + 9$, is sometimes called a "quartic"
- a fifth-degree polynomial, such as $2x^5$ or $x^5 – 4x^3 – x + 7$, is sometimes called a "quintic"

Evaluating a polynomial is the same as evaluating any other expression. Simply plug in the given value of *x*, and figure out what the value of the polynomial is.

Example:

Evaluate $2x^3 – x^2 – 4x + 2$ at $x = –3$

Solution: Plug in "–3" for the "*x*" and simplify

$2(–3)^3 – (–3)^2 – 4(–3) + 2$
$= 2(–27) – (9) + 12 + 2$
$= –54 – 9 + 14$
$= –63 + 14$
= **–49**

Operations with Polynomials

To add or subtract polynomials, we combine like terms. **Like terms** are terms which have the same variables raised to the same exponents.

$4x$ and 3	NOT like terms	The second term has no variable
$4x$ and $3y$	NOT like terms	The second term now has a variable, but it doesn't match the variable of the first term

$4x$ and $3x^2$	NOT like terms	The second term now has the same variable, but the degree is different
$4x$ and $3x$	LIKE TERMS	Now the variables match and the degrees match

Adding and Subtracting polynomials:
To add or subtract polynomials, we collect like terms.

Examples:

1. Add the polynomials: $(10x^2 – 7x + 5) + (2x^2 + 2x – 15)$

   Solution:
   $(10x^2 – 7x + 5) + (2x^2 + 2x – 15)$ Remove the parentheses
   $= 10x^2 – 7x + 5 + 2x^2 + 2x – 15$ Collect like terms
   $= 12x^2 – 5x – 10$

2. Subtract the polynomials: $(3x^2y^2 – 7xy + 2x) – (x^2y^2 + 8xy – 5y)$

   Solution:
   $(3x^2y^2 – 7xy + 2x) – (x^2y^2 + 8xy – 5y)$ Distribute the negative through the parentheses
   $= 3x^2y^2 – 7xy + 2x – x^2y^2 – 8xy + 5y$ Collect like terms
   $= 2x^2y^2 – 15xy +2x + 5y$

Multiplying Polynomials:
To multiply two polynomials, all terms of the first polynomial must be multiplied by all terms of the second polynomial. Remember when multiplying variables with the same base the exponents are added.

Examples:

1. Multiply: $6x^4y(2x^2y^2 + 7x)$

   Solution:
   $6x^4y(2x^2y^2 + 7x)$ Distribute
   $= 12x^6y^3 + 7x^5y$

2. Multiply: $(2x + 5)(3x + 4)$

   Solution:
   To multiply two binomials, we multiply the first terms (F), the outside terms (O), the inside terms (I), and then the last terms (L). This method is called FOIL.
   $(2x + 5)(3x + 4)$ Multiply (FOIL)
   $= 6x^2 + 8x + 15x + 20$ Collect like terms
   $= 6x^2 + 23x + 20$

3. Multiply: $(3x^2 + 7x - 1)(2x^2 + 4x + 5)$

Solution:

Each term of one polynomial must be multiplied by each term of the other:

$(3x^2 + 7x - 1)(2x^2 + 4x + 5)$	
$= 6x^4 + 12x^3 + 15x^2$	First term times the second polynomial
$+ 14x^3 + 28x^2 + 35x$	Second term times the second polynomial
$- 2x^2 - 4x - 5$	Third term times the second polynomial
$= 6x^4 + 26x^3 + 41x^2 + 31x - 5$	Collect like terms

Dividing Polynomials:

If dividing by a monomial, split the problem into separate fractions with the common denominator and then simplify each separate fraction.

Example:

Divide: $\frac{4x^3-2x^2+5}{2x}$

Solution:

$\frac{4x^3-2x^2+5}{2x}$ Separate into 3 fractions

$= \frac{4x^3}{2x} - \frac{2x^2}{2x} + \frac{5}{2x}$ Simplify each term

$= 2x^2 - x + \frac{5}{2x}$

If dividing by a binomial, then long division is used to divide the polynomials. Long division for polynomials is similar to long division of numbers.

Steps for long division:

1. Rewrite as a division problem.
2. Look at the first term of each polynomial. Divide the first term of the dividend by the first term of the divisor. Write this answer on top of the division sign.
3. Multiply the term you just wrote on top by the divisor.
4. Subtract by adding the opposite.
5. Bring the next term of the dividend down and repeat the steps 2-5.
6. The remainder is written as a fraction with the divisor as the denominator.

Examples:

1. Divide: $\frac{x^2-9x-10}{x+1}$

Step 1: Ignore the other terms and look just at the leading *x* of the divisor and the leading x^2 of the dividend.	$x+1\overline{)x^2 \quad -9x-10}$
Divide the leading x^2 inside by the leading *x* in front. This is *x*. So put an *x* on top:	$\begin{array}{r} x \qquad\qquad \\ x+1\overline{)x^2 \quad -9x-10} \end{array}$
Now take that *x*, and multiply it through the divisor, $x+1$. First, I multiply the *x* (on top) by the *x* (on the "side"), and carry the x^2 underneath:	$\begin{array}{l} \qquad\quad x \\ x+1\overline{)x^2 \quad -9x-10} \\ \qquad\ x^2 \end{array}$
Then I'll multiply the *x* (on top) by the 1 (on the "side"), and carry the $1x$ underneath:	$\begin{array}{l} \qquad\quad x \\ x+1\overline{)x^2 \quad -9x-10} \\ \qquad\ x^2 \quad +1x \end{array}$
Then draw the "equals" bar and subtract. To subtract the polynomials, *change all the signs* in the second line...	$\begin{array}{l} \qquad\quad x \\ x+1\overline{)x^2 \quad -9x-10} \\ \qquad \underline{-x^2 \quad \overset{-}{+}1x \qquad} \end{array}$
...and then I add down. The first term (the x^2) will cancel out:	$\begin{array}{l} \qquad\quad x \\ x+1\overline{)x^2 \quad -9x-10} \\ \qquad \underline{-x^2 \quad \overset{-}{+}1x \qquad} \\ \qquad\qquad -10x \end{array}$
Bring down the next term from the dividend:	$\begin{array}{l} \qquad\quad x \\ x+1\overline{)x^2 \quad -9x-10} \\ \qquad \underline{-x^2 \quad \overset{-}{+}1x \qquad} \\ \qquad\qquad -10x-10 \end{array}$
Now look at the *x* from the divisor and the new leading term, the $-10x$, in the bottom line of the division. Dividing the $-10x$ by the *x*, we get -10, so that goes on top:	$\begin{array}{l} \qquad\quad x \ -10 \\ x+1\overline{)x^2 \quad -9x-10} \\ \qquad \underline{-x^2 \quad \overset{-}{+}1x \qquad} \\ \qquad\qquad -10x-10 \end{array}$
Now multiply the -10 (on top) by the leading *x* (on the "side"), and carry the $-10x$ to the bottom:	$\begin{array}{l} \qquad\quad x \ -10 \\ x+1\overline{)x^2 \quad -9x-10} \\ \qquad \underline{-x^2 \quad \overset{-}{+}1x \qquad} \\ \qquad\qquad -10x-10 \\ \qquad\qquad -10x \end{array}$

...and multiply the -10 (on top) by the 1 (on the "side"), and carry the -10 to the bottom:	$\begin{array}{r} x - 10 \\ x+1\overline{)x^2 \; - 9x - 10} \\ -x^2 \; \overset{-}{+}1x \\ \hline -10x - 10 \\ -10x - 10 \end{array}$
Draw the equals bar, and *change the signs* on all the terms in the bottom row:	$\begin{array}{r} x - 10 \\ x+1\overline{)x^2 \; - 9x - 10} \\ -x^2 \; \overset{-}{+}1x \\ \hline -10x - 10 \\ +10x + 10 \\ \hline \end{array}$
Then add down:	$\begin{array}{r} x - 10 \\ x+1\overline{)x^2 \; - 9x - 10} \\ -x^2 \; \overset{-}{+}1x \\ \hline -10x - 10 \\ +10x + 10 \\ \hline 0 \end{array}$

Then the solution to this division is: ***x* – 10**

2. Divide: $\frac{2x^2+5x-7}{x+3}$

Step 1: Ignore the other terms and look just at the leading *x* of the divisor and the leading $2x^2$ of the dividend.	$x+3\;\overline{)2x^2+5x-7}$
Divide the leading $2x^2$ inside by the leading *x* in front. This is 2*x*. So put an *x* on top:	$\begin{array}{r} 2x \\ x+3\;\overline{)2x^2+5x-7} \end{array}$
Now take that *x*, and multiply it through the divisor, *x* + 3.	$\begin{array}{r} 2x \\ x+3\;\overline{)2x^2+5x-7} \\ 2x^2+6x \end{array}$
Then draw the "equals" bar and subtract. To subtract the polynomials, *change all the signs* in the second line...	$\begin{array}{r} 2x \\ x+3\;\overline{)2x^2+5x-7} \\ -2x^2-6x \\ \hline \end{array}$
...and then I add down. The first term (the $2x^2$) will cancel out. Bring down the next term from the dividend.	$\begin{array}{r} 2x \\ x+3\;\overline{)2x^2+5x-7} \\ -2x^2-6x \\ \hline -x-7 \end{array}$

Now look at the *x* from the divisor and the new leading term, the –*x*, in the bottom line of the division. Dividing the –*x* by the *x*, we get -1, so that goes on top:	$\begin{array}{r} 2x \;-\; 1 \\ x+3\;\overline{)2x^2+5x-7} \\ -2x^2-6x \\ \hline -x-7 \end{array}$
Now multiply the –10 (on top) by the leading *x* (on the "side"), and carry the –10*x* to the bottom:	$\begin{array}{r} 2x \;-\; 1 \\ x+3\;\overline{)2x^2+5x-7} \\ -2x^2-6x \\ \hline -x-7 \\ \underline{-x-3} \end{array}$
Change the signs on all the terms in the bottom row to subtract. Then add down.	$\begin{array}{r} 2x \;-\; 1 \\ x+3\;\overline{)2x^2+5x-7} \\ -2x^2-6x \\ \hline -x-7 \\ \underline{+x+3} \\ -4 \end{array}$

The solution is $2x - 1 + \frac{-4}{x+3}$.

Synthetic division

Synthetic division is a shortcut that can be used to divide a polynomial f(x) by (x - k). You should get the same result with synthetic division as you would with long division.

Steps for synthetic division:

1. Identify the value of k.
2. Put k in a box and then write the coefficients of the polynomial f(x) to its right. Be sure to put in a zero for any degree term that is missing from the polynomial.
3. Copy the first coefficient into the third row. Multiply this coefficient by k and write the result below the next coefficient of f(x) in the second row. Add the first and second row and put the result into the third row. Now repeat the process.
4. The last number in the third row is the remainder. The other numbers are the coefficients of the quotient in descending powers.

Examples:

1. Divide $\frac{2x^2+5x-7}{x+3}$ using synthetic division.

   k = -3 so place the -3 in a box, then write the coefficients of $2x^2 + 5x – 7$. Bring the 2 down to the third row. Now, multiply 2 by -3 and put the result (-6) into row 2. Now add 5 + -6 and put the result (-1) in the third row. Multiply -1 by -3 and put the result in the second row. Add.

$-3\rfloor$	2	5	-7
		-6	3
	2	-1	-4

The remainder is -4 and the other numbers in row 3 are the coefficients of the quotient. So, the answer is $2x - 1 + \frac{-4}{x+3}$. Notice that this was the same problem as in the previous example.

2. Use synthetic division to divide $3x^3 - 7x^2 + 5$ by $x – 4$.

   k = 4. Notice that the x term is missing so its coefficient is 0.

$4\rfloor$	3	-7	0	5
		12	20	80
	3	5	20	85

The remainder is 85 and the quotient is $3x^2 + 5x + 20$ so the result is $3x^2 + 5x + 20 + \frac{85}{x-4}$.

Homework:

1. Which of the following is **not** a polynomial?

   A. $5x^3 - x/2$ B. $3xy - 4yz + 2xz$

   C. $52x^3 - 19y^5$ D. $\frac{7}{x-3}$

2. Which of the following is a polynomial?

   A. $3x^2 - 3x^{½} + 2y^{2½}$ B. $-7x^3y^{-1}$

   C. $5x^3 - 3xy^2 + 8y^4 – 3$ D. $\frac{2}{x+5}$

3. How many terms does the polynomial $2x^3 - 4x + 7$ have?
4. How many terms does the polynomial $3x^3y + 6x^2y^3 + 2x^3 + 4x^2y^2$ have?
5. How many terms does the polynomial $-0.5x^3y + 5x^2y^3$ have?
6. For the following polynomials, list the terms, the coefficient of each term, the degree of each term, and the degree of the polynomial.

   A. $2x^3 - 4x^2 + 7$

   B. $3x^3y + 6x^2y^3 + 2x^3 + 4x^2y^2$

   C. $-0.5x^3y + 5x^2y^3$

   D. $7y + 2$

E. ½ x^4y^8

7. Are the following monomials, binomials, trinomials, or none of these?

A. $2x^3 - 4x^2 + 7$

B. $3x^3y + 6x^2y^3 + 2x^3 + 4x^2y^2$

C. $-0.5x^3y + 5x^2y^3$

D. $7y + 2$

E. ½ x^4y^8

8. Find the degree of the polynomial and indicate whether the polynomial is a monomial, binomial, trinomial, or none of these: $10x^2 – 8x^4$

9. Find the degree of the polynomial and indicate whether the polynomial is a monomial, binomial, trinomial, or none of these: -20

10. What is the degree of the polynomial $5x^3 - 8x + 3x^5 + 4x^2 - 7x^4 + 1$? Put the polynomial in standard form.

11. Write $7x^2 - 4x^5 + 12 + 3x^3 - 2x + 2x^4$ in standard form.

Add or subtract the following:

12. A. $(6x + 7) + (9x – 2)$ B. $(8x – 11) + (-2x – 7)$

13. A. $(3x^2 – 5x + 1) + (x^2 – 9x)$ B. $(4x^2 – 9x + 5) + (4x – 9)$

14. A. $(13x^2 + 7x + 1) – (x^2 – 4x – 1)$ B. $(6x^2 – x + 1) – (x^2 + 9x + 1)$

15. A. $(13x^2y + 7xy + 11y^2) – (x^2y + 3xy – y^2)$ B. $(7x^2y + 2y + 10y^2) + (2x^2y + 3x – 6y^2)$

Multiply the following:

16. $4x^3(2x^2 + 5x)$

17. $5x(4x^2 + x – 8)$

18. $(3x – 7)(2x + 1)$

19. $(x + 4)(2x^2 + 4)$

20. $(2x + 3)(5x – 9)$

21. $(3x^2 – 1)(2x^2 + 5)$

22. $(3x + 4) (x^2 + 2x – 1)$

23. $(2x + 5)(3x^2 – x + 2)$

24. $(2x^2 + x – 4)(5x^2 – 3x + 2)$

25. $(3x – 2y)(6x + y)$

Divide the following:

26. $\frac{5x^2+20x+10}{5x}$

27. $\frac{4x^2+20x+10}{2x^2}$

28. $\frac{6x^3+20x^2+12x}{3x}$

29. $\frac{x^3+20x^2+12x-1}{2x^2}$

Divide the following using long division.

30. $(x^2 + 6x – 2) \div (x + 2)$

31. $(3x^2 + 5x + 2) \div (x – 1)$

32. $(4x^2 – 2x + 3) \div (x + 3)$

33. $(6x^2 + x – 11) \div (x – 2)$

34. $(x^3 - 3x^2 – 2x + 3) \div (x + 1)$

Divide the following using synthetic division.

35. $(x^2 + 5x + 2) \div (x + 2)$

36. $(x^3 - 3x^2 + 5x + 2) \div (x – 1)$

37. $(5x^2 – 2x + 3) \div (x + 3)$

38. $(3x^2 + x – 9) \div (x – 2)$

38. $(2x^3 – 7x + 3) \div (x + 1)$

Evaluate the polynomials:

39. For $f(x) = -12x^2 - 11x - 18$, find $f(-1)$.

40. For $f(x) = 4x^2 + 3x - 8$, find $f(-1)$.

41. For $f(x) = 3x^3 + 3x^2 + 2x + 8$, find $f(-4)$.

42. For $f(x) = 5x^3 - 12x^2 - x + 11$, find $f(3)$.

4.7 Complex Numbers

A **complex number** is a number that can be put in the form *a* + *bi*, where *a* and *b* are real numbers and *i* is called the imaginary unit, where $i^2 = -1$. In this expression, *a* is called the real part and *b* the imaginary part of the complex number. A complex number whose real part is zero is said to be purely imaginary, whereas a complex number whose imaginary part is zero is a real number. In this way the complex numbers contain the ordinary real numbers while extending them in order to solve problems that cannot be solved with only real numbers.

Complex numbers are used in many scientific fields, including engineering, electromagnetism, quantum physics, and applied mathematics, such as chaos theory. Italian mathematician Gerolamo Cardano is the first known to have introduced complex numbers. He called them "fictitious", during his attempts to find solutions to cubic equations in the 16th century.

Definition: The **imaginary number** $i = \sqrt{-1}$ (or $i^2 = -1$).

Examples:

1. Simplify $\sqrt{-25}$.

   Solution: $\sqrt{-25} = \sqrt{25}\sqrt{-1} = 5i$

2. Simplify $\sqrt{-100}$.

   Solution: $\sqrt{-100} = \sqrt{100}\sqrt{-1} = 10i$

Definition: The **standard form of a complex number** is a + bi where a is the real part and bi is the imaginary part.

Complex numbers in the form a + b*i* can be added, subtracted, multiplied and divided.

To add or subtract complex numbers, we combine the real parts together and the imaginary parts together.

Examples:

1. Add: (9 + 2i) + (5 + 2i)

   Solution: (9 + 2i) + (5 + 2i) = (9 + 5) + (2i + 2i) = 14 + 4i

2. Add: (7 + 5i) + (11 – i)

   Solution: (7 + 5i) + (11 – i) = (7 + 11) + (5i – i) = 18 + 4i

3. Subtract: (-3 + 19i) – (4 + 6i)

   Solution: (-3 – 4) + (19i – 6i) = -7 + 13i

Multiplying Complex Numbers

Complex numbers are multiplied in the same way as polynomials. All terms of one complex number must be multiplied by all terms of the other. Remember that $i^2 = -1$ and use this to simplify the result.

Examples:

1. Multiply 5i(2 + 3i).

   Solution:

$5i(2 + 3i) = 10i + 15i^2$	Using the distributive property
$= 10i + 15(-1)$	Replacing i^2 with -1
$=-15 + 10i$	Writing in standard form

2. Multiply (9 – i)(2 + 5i).

   Solution:

$(9 - i)(2 + 5i)$	FOIL
$= 18 + 45i - 2i - 5i^2$	Collecting like terms and replacing i^2 with -1
$= 18 + 43i - 5(-1)$	Collecting like terms
$=23 + 43i$	

3. Multipy (2 + 7i)(2-7i).

   Solution:

$(2 + 7i)(2-7i)$	FOIL
$= 4 - 14i + 14i - 49i^2$	Combine like terms and replacing i^2 with -1
$= 4 - 49(-1)$	
$= 53$	

Notice in example 3 above that we multiplied two complex numbers and the solution was a real number. Complex numbers of the form a + bi and a – bi are called **complex conjugates**. When complex conjugates are multiplied, the result is a real number. These are used in the division of complex numbers. For instance, 4 + i and 4-i are complex conjugates and so are 3i and -3i.

Dividing Complex Numbers

When dividing complex numbers, the result should have no complex numbers in the denominator. To divide complex numbers, multiply the numerator and denominator by the complex conjugate of the denominator and simplify. Be sure to put the final answer in standard form.

Examples:

1. Divide: (6 – 7i) ÷ (2i)

   Solution:
   The complex conjugate of 2i is -2i.

   $\frac{6-7i}{2i}$ Rewrite as a fraction

   $= \frac{6-7i}{2i} \cdot \frac{-2i}{-2i}$ Multiply by the complex conjugate

   $= \frac{-12i+14i^2}{-4i^2}$ Replace i^2 with -1

   $= \frac{-12i+14(-1)}{-4(-1)}$ Simplify and write in standard form

   $= \frac{-12i-14}{4} = -\frac{7}{2} - 3i$

2. Divide: (1 + 3i) ÷ (2 – i)

   Solution:
   The complex conjugate of the denominator is 2 + i.

   $\frac{1+3i}{2-i}$ Rewrite as a fraction

   $= \frac{1+3i}{2-i} \cdot \frac{2+i}{2+i}$ Multiply by the complex conjugate 2 + i

   $= \frac{2+i+6i+3i^2}{4+2i-2i-i^2}$ Simplify and replace i^2 by -1

   $= \frac{2+7i+3(-1)}{4-(-1)}$ Simplify and write in standard form

   $= \frac{-1+7i}{5} = -\frac{1}{5} + \frac{7}{5}i$

Powers of i

The imaginary number, *i*, when raised to a power can be simplified to one of 4 results: 1, *i*, -1, or –*i*.
$i = i$
$i^2 = -1$
$i^3 = i^2 \cdot i = -i$
$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$
$i^5 = i^4 \cdot i = i$
and the pattern continues. Since $i^{4m} = 1$, where m is any whole number, then to simplify powers of *i*, rewrite the problem as *i* to the 4^{th} power (or a multiple of 4) times the remaining power needed. Then, simplify as shown above.

Examples:

Simplify the following:

1. i^{11}

   We rewrite with an exponent which is the largest multiple of 4 less than 11. Recall $i^8 = i^{4\cdot 2} = 1$.

   $i^{11} = i^8 \cdot i^3 = i^3 = -i$

2. i^{22}

   Rewriting: $i^{22} = i^{20} \cdot i^2 = i^2 = -1$

3. i^{100}

   Since 100 is a multiple of 4, $i^{100} = 1$.

4. i^9

   Rewriting: $i^9 = i^8 \cdot i = i$

Homework:

Perform the indicated operation and simplify:

1. $\sqrt{-81}$
2. $\sqrt{-9}$
3. $\sqrt{-25} \cdot \sqrt{-16}$
4. $\sqrt{-49} + \sqrt{-121}$
5. (2 + 4i) + (-7 – 9i)
6. (4 + i) + (3 – 9i)
7. (3 + 8i) – (8 + 2i)
8. (5 – 9i) – (1 – 7i)
9. (5 + 2i)(4 – 3i)
10. (10 +3i)(5 – i)
11. (8 + 3i)(2 + 9i)
12. (6 – 5i)(2 – 3i)
13. 5i(8 – 2i)
14. 2i(4 + 5i)
15. (3 + 5i)(3 – 5i)
16. (7 – 2i)(7 + 2i)
17. (3 – 9i) ÷ (1 + 2i)
18. (4 + 7i) ÷ (3 – 4i)
19. (1 + 6i) ÷ (1 - 3i)
20. (3 – 4i) ÷ (5 - 2i)

Simplify the powers of i.

21. i^{31}
22. i^{17}
23. i^{50}
24. i^{16}